

"Is there place for free will in mathematics?"

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Let  $M_\alpha$  be sets indexed by  $\alpha \in A$

The axiom of choice: We can form a new set, choosing from each  $M_\alpha$  one element  $m_\alpha \in M_\alpha$

There exists a "choice function"

$$\varphi : \alpha \in A \rightarrow m_\alpha \in M_\alpha$$

which assigns to each index  $\alpha \in A$  an element  $\varphi(\alpha) := m_\alpha$  belonging to  $M_\alpha$

This axiom stands at the very basis of many branches of mathematics

- i) **Theorem:** Each infinite set contains a countable subset

Proof: Let  $M$  be an infinite set. Choose an element  $a_1 \in M$ . Next choose another element  $a_2 \in M \setminus \{a_1\}$ , next choose  $a_3 \in M \setminus \{a_1, a_2\}$  and so on... This process will never stop because  $M$  is infinite.

We have found the countable subset

$$A := \{a_1, a_2, a_3, \dots\} \subset M$$

**Corollary:** Among infinite sets the countable ones are the "smallest"

Many other examples:

- ii) Cartesian product of infinite sets
- iii) Existence of a basis in each infinite vector space
- iv) Hahn-Banach theorem, functional analysis, measure theory, topology . . .

Renouncing to the axiom of choice would mean renounce to fundamentals results of modern mathematics

At the end of 19 century people tried to avoid this axiom

Why?

- 1) The choice function is, in general, a not mechanical rule, it can not be implemented by a computer!
- 2) The paradoxes arising in set theory: If we define sets in a too freely way we reach absurdities!

Cantor-Russel Paradox The set  $\Omega$  of all the sets  $X$  which does not contain itself as an element

$$\Omega := \{X : X \notin X\}$$

Which kind of set  $\Omega$  is?

If

$$\Omega \in \Omega \implies \Omega \notin \Omega$$

If

$$\Omega \notin \Omega \implies \Omega \in \Omega$$

We can assume a very **conservative** attitude:

We are allowed to consider only sets which are defined by means of purely mechanical, algorithmic rules (recursively enumerable sets)

But also this point of view does not survive:

**The Turing Theorem:** there exists a **NON** recursively enumerable set

Proof. The set of

$\{n \in \mathbb{N} \mid \text{the Turing machine } f_n \text{ has infinite outputs}\}$

is not generated by an algorithm!! It is a so called Cantor Diagonal argument. ■

Therefore we must allow to define sets by means of properties which avoid two problems

- 1) Avoid the absurdities of Russel paradoxes
- 2) Allow for defining sets in a highly non algorithmic way (allowing for intuitive insight of the mind!)

These tasks are provided by the axiomatic system of set theory of Zermelo and Fraenkel (ZF)

In particular the "Axiom of replacement" allows to form sets, defined by properties, in a highly non-algorithmic way (it takes into account insights of the mind!) but (presumably!) does not lead to any contradiction (because the cardinality of the new sets is not greater than the cardinality of the previous ones)

Therefore people thought it was maybe possible to prove the Axiom of Choice (AC) in this powerful system

- (AC) can not be disproved in (ZF) (Gödel, 1938)

It was very reassuring because people used very freely (AC) in mathematical proofs

- (AC) can not be proved in (ZF) (Cohen, 1964)

$\Rightarrow$  (AC) is undecidable in (ZF)!!!

## I Philosophical Conclusion

The Zermelo-Fraenkel system is a very natural universe model of set theory:

”...all our intuition comes from our belief in the natural, almost physical, model of the mathematical universe. Indeed with regard to ZF it is hard to conceive of any other model.” P. Cohen

⇒ The Axiom of Choice in mathematics points to an independent faculty of our mind, -freedom of choice, free will- distinct by the non algorithmic insights which we concluded by Gödel-Turing Theorem

**A very important remark:** dealing with countable sets, there is no need of the axiom of choice! The choice function can be constructed by the induction principle!

The real numbers  $\mathbf{R}$  are not countable, this is again the diagonal argument of Cantor.

Now, the **Computable Numbers** -numbers which are generated by a computer- are a countable infinity, as many as the countable finite programs that can be written.

$\implies$  there exists "much more" real numbers than computable numbers

how can we write a real number which is not computable??

00101011000111100011111000000110101010110....

is computable? which is the next digit? By definition of non-computable number there is NO algorithmic rule!

To write down an irrational real number not computable requires a great intelligence!

## II Philosophical Conclusion

The **Undeterminacy** is required to allow a **Free choice**. It is the possibility for an intelligent mind to freely operate

The concept of **Randomness** is **NOT** opposed to **Purpose**, it does not necessarily mean a "blind process"