Proof of Turing theorem

(*Taken from A. Suarez in "Mathematical undecidability…" , 1993*)

By contradiction.

Let us suppose that there exists an algorithm U whose output is the list of all these S.
Therefore it is possible to make the list of all the sets S generated by an algorithm (recursively enumerable sets) which have infinitely many elements:

$\mathbf{S_1} = \{a_{1;1}, a_{1;2}, a_{1;3} \dots a_{1;n} \dots\}$

$\mathbf{S_2} = \{a_{2;1}, a_{2;2}, a_{2;3} \dots a_{2;n} \dots\}$

$\mathbf{S_3} = \{a_{3;1}, a_{3;2}, a_{3;3} \dots a_{3;n} \dots\}$

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$\mathbf{S_n} = \{a_{n;1}, a_{n;2}, a_{n;3} \dots a_{n;n} \dots\}$

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Let us construct the set $\mathbf{T} = \{b_1, b_2, b_3 \dots b_n \dots\}$

according to the following rule:

$b_1 = a_{1;1} + 1$

$b_2 = a_{2;2} + 1$ if $a_{2;2} \geq b_1$ $\quad$ $b_2 = b_1 + 1$ if $a_{2;2} < b_1$

$b_3 = a_{3;3} + 1$ if $a_{3;3} \geq b_2$ $\quad$ $b_3 = b_2 + 1$ if $a_{3;3} < b_2$

…

$b_n = a_{n;n} + 1$ if $a_{n;n} \geq b_{n-1}$ $\quad$ $b_n = b_{n-1} + 1$ if $a_{n;n} < b_{n-1}$

The set T contains an increasing sequence of infinitely many natural numbers, generated by an algorithm (recursively enumerable) , which is not contained in the previous list, i.e. T differs from S_i for every i.

Conclusion:

On the one hand, T is an infinite recursively enumerable set , i.e. it is generated by an algorithm, and therefore it should belong to the previous list.

On the other hand, we have proved that the set T is not contained in the previous list.

Contradiction!

Hence the algorithm U does not exist.