Proof of Gödel theorem starting from Turing theorem

First of all, let us prove that the set P of the formally provable propositions in a formal system F of the arithmetic is recursively enumerable.

Definition of recursively enumerable set:
a set A is recursively enumerable if it can be generated by an algorithm.

Let us consider all the propositions of the formal system and let's sort them like this:

Propositions of length 1

O

O

.

.

O

Propositions of length 2

OO

OO

.

.

OO

Propositions of length 3

OOO

OOO

.

.

OOO ………..

WELL FORMED PROPOSITIONS

Deciding whether or not a proposition is well formed is a matter of computing, in fact if the established syntax rules are respected within the proposition, then this proposition is well formed.

Hence there exists an algorithm which is able to generate all the well formed propositions and we will consider only these ones in the following.

FORMAL PROOFS

Deciding whether or not a proposition is a formal proof is a matter of computing, in fact if the rules of deduction, i.e. the axioms of the formal system, are respected within the proposition, then this proposition is a formal proof. Hence there exists an algorithm which is able to provide the list of all the propositions that are formal proofs.

FORMALLY PROVABLE PROPOSITIONS

Formal proofs “end up” with a formally provable proposition which is the last “line” of the formal proof.

So, it is possible to “take out” from each formal proof its corresponding formally provable proposition. Even this procedure is algorithmic.

Hence there exists an algorithm which is able to provide the list of all the formally provable propositions.

That is to say, for definition of recursively enumerable set:

The set P of all the formally provable propositions of the formal system F is a recursively enumerable set.

FORMALLY DISPROVABLE PROPOSITIONS

Even the set C of all the formally disprovable propositions is a recursively enumerable set:

Let’s consider the list of the formally provable propositions A.

Let’s construct the list of the propositions ~A; these ones are all formally disprovable as the propositions ~(~A) = A are formally provable.

Inversely each proposition B that is formally disprovable belongs to the list as B = ~(~B) and ~B is one of the A propositions since it is formally provable. Hence:

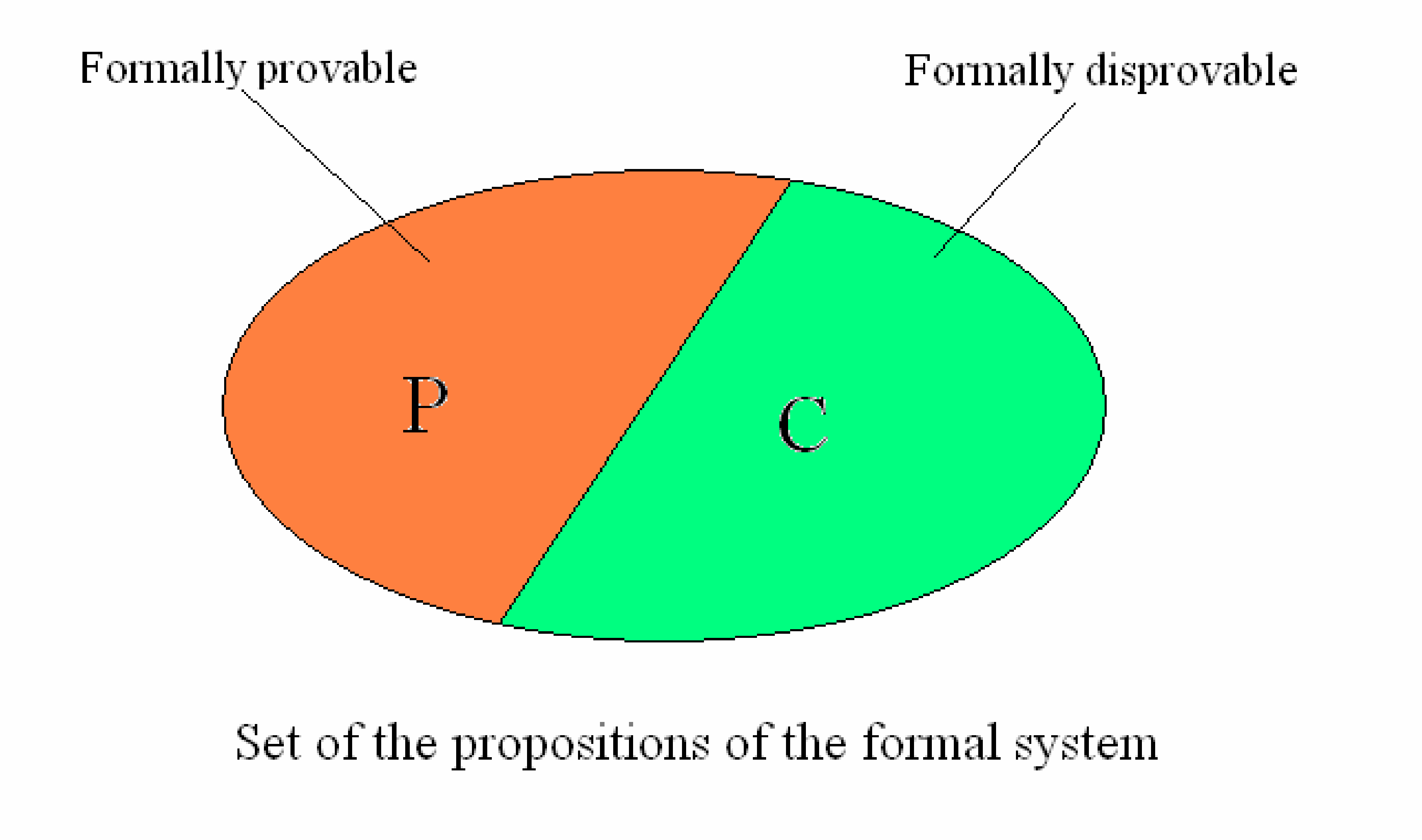
<u>The set C of the formally disprovable propositions of the formal system F is a recursively enumerable set.</u>

Starting from Turing theorem it is possible to show that the set P is not recursive.

Once it is proved that the set P is not recursive, Gödel theorem can be inferred.

Definition of recursive set:
a set A is recursive if it is recursively enumerable and if even its complement set is recursively enumerable.

Let's suppose ab absurdo that Gödel theorem is false and so that all the propositions of the formal system are either formally provable or formally disprovable.



That means that the complement set of the set P coincides with the set C.

As P and C are recursively enumerable the set P is recursively enumerable and even its complement set C is recursively enumerable . Hence P is recursive.

This is absurd, because from Turing's theorem it is possible to prove that P is NOT recursive.

Hence:

Once it is proved that the set P is not recursive, Gödel's theorem can be inferred.