"On the axiom of Choice"

London 2007, "Does God Play dice?"

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Let M_α be sets indexed by $\alpha \in A$

The axiom of choice: We can form a set, choosing from each set M_α one element $m_\alpha \in M_\alpha$

There exists a "choice function"

$$\varphi : \alpha \in A \to m_\alpha \in M_\alpha$$

which assigns to each index $\alpha \in A$ an element $\varphi(\alpha) := m_\alpha$ belonging to the corresponding set M_α

This axiom stands at the very basis of many branches of mathematics

- i) Theorem: Each infinite set contains a countable subset

  Proof: Let M be an infinite set. Choose an element $a_1 \in M$. Next choose an element $a_2 \in M \setminus \{a_1\}$, next choose $a_3 \in M \setminus \{a_1, a_2\}$ and so on... This process will never stops because M is infinite.

  We have found the countable subset

  $$A := \{a_1, a_2, a_3, \ldots\} \subset M$$

  Corollary: Among infinite sets the countable ones are the "smallest"

- ii)Cartesian product of sets

  The cartesian product two finite sets, e.g.

  $$M_1 := \{a_1, b_1, c_1\} \qquad M_2 := \{a_2, b_2\}$$

  is the set of 6 pairs $M_1 \times M_2 :=$

  $\{(a_1, a_2), (a_1, b_2), (b_1, a_2), (b_1, b_2), (c_1, a_2), (c_1, b_2)\}$

  where the first place contains an element of the set M_1 and the second place an element of the set M_2

How to define the cartesian product of infinitely many infinite sets

$$\Pi_{\alpha \in A} M_\alpha = \ldots \times M_\alpha \times \ldots \times M_{\alpha'} \times \ldots ?$$

This is the set of all the functions

$$A \ni \alpha \to \varphi(\alpha) \in M_\alpha$$

We recognize the use of the axiom of choice even to state very basic concepts in mathematics and physics

Many other examples:

- iii) All cardinals numbers are comparables

- iv) Existence of a basis in each infinite vector space

- v) Hahn-Banach theorem, essential in functional analysis, measure theory, topology...

Renouncing to the axiom of choice would mean renounce to fundamentals results of modern mathematics

At the end of 19 century people tried to avoid this axiom

Why?

- 1) The choice function is, in general, a not mechanical rule, it can not be implemented by a computer!

  $\Longrightarrow$ It is clarified the concept of algorithm, Turing machine, recursive functions, ...

- 2) The paradoxes arising in set theory: If we define sets in a too freely way we reach absurdities!

Cantor-Russel Paradox The set Ω of all the sets X which does not contain itself as an element

Example 1) The set $X := \{$The chairs in this room$\}$. The set X is not a chair in this room, hence $X \notin X$ ("ordinary set")

Example 2) The set $X := \{$Everything which is not this computer$\}$. The set X itself is not a computer and then $X \in X$ ("extraordinary set")

$$\Omega := \{X : X \notin X\}$$

Which kind of set Ω is? Is Ω ordinary or extraordinary?

If

$$\Omega \in \Omega \quad \Longrightarrow \quad \Omega \notin \Omega$$

If

$$\Omega \notin \Omega \quad \Longrightarrow \quad \Omega \in \Omega$$

We can assume a very conservative attitude:

We are allowed to consider only sets which are defined by means of purely mechanical, algorithmic rules (recursively enumerable sets)

But also this point of view does not survive:

The Turing Theorem: there exists a NON recursively enumerable set

Proof. The set of

$\{n \in \mathbb{N} \mid$ the Turing machine A(n) does not stop$\}$

is not generated by an algoritmh!! ∎

Therefore we must allow to define sets by means of properties which avoid two problems

- 1) Avoid the absurdities of Russel paradoxes

- 2) Allow for defining sets in a highly non algoritmic way (allowing for intuitive insight of the mind!)

These tasks are provided by the axiomatic system of set theory of Zermelo and Fraenkel (ZF)

In particular the "Axiom of replacement" allows to form sets, defined by properties, in a highly non-algorithmic way (it takes into account insights of the mind!) but (presumably!) does not lead to any contradiction (because the cardinality of the new sets is not greater than the cardinality of the previous ones)

Therefore people thought it was maybe possible to prove the Axiom of Choice (AC) in this powerful system

- (AC) can not be disproved in (ZF) (Gödel, 1938)

  It was very reassuring because people used very freely (AC) in mathematical proofs

- (AC) can not be proved in (ZF) (Cohen, 1964)

$\Longrightarrow (AC)$ *is undecidable in* (ZF)!!!

I Philosophical Conclusion

The Zermelo-Fraenkel system is a very natural universe model of set theory:

"...all our intuition comes from our belief in the natural, almost physical, model of the mathematical universe. Indeed with regard to ZF it is hard to conceive of any other model." P. Cohen

$\Longrightarrow$ The Axiom of Choice in mathematics points to an independent faculty of our mind, -freedom of choice, free will- distinct by the non algorithmic insights which we concluded by Gödel-Turing Theorem

The real numbers $\mathbb{R}$ are not countable, this is again the diagonal argument of Cantor.

Now, the Computable Numbers -numbers which are generated by a computer- are a countable infinity, as many as the countable finite programs that can be written.

$\Longrightarrow$ there exists "much more" real numbers than computable numbers

how can we write a real number which is not computable??

$\mathbb{R}$ is exacty the cartesian product set

$$\mathbb{R} := \{0, 1, 2, \ldots, 9\}^{\mathbb{N}} := \{\mathbb{N} \rightarrow \{0, 1, 2, \ldots, 9\}\}$$

(in basis 10) We can clearly do this with the axiom of choice

II Philosophical Conclusion

The undeterminacy is required to allow a free choice, being the possibility that an intelligent mind can freely operate

The concept of Randomness is NOT opposed to purpose, it does not necessarily mean a "blind process"

To write down an irrational real number not computable requires a great intelligence!